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FUZZY METRIC SPACES AND EXTENSION OF FIXED POINT THEOREMS THROUGH RATIONAL INEQUALITY WITH APPLICATIONS

SURENDRA KUMAR TIWARI[1] AND RANU AGRAWAL[2]
[1,2] Department of Mathematics,
Dr. C.V. Raman University, Kargi Road, Kota,
Bilaspur (C.G.)-India

Abstract

In 1965, Zadeh introduced fuzzy set theory, which has since led to significant advancements in developing fuzzy analogs of classical theories. This theory has found a wide range of applications across various fields. One of the key advantages of fuzzy mathematics is its practical use beyond pure mathematics, including in areas such as quantum particle physics, as explored by El Naschie in 2004. This manuscript aims to present and investigate fixed point theorems related to contractive mappings, as well as to examine rational expressions of specific theoretical outcomes concerning fuzzy metric spaces. Our results extend, generalize, and unify several well-established findings in the existing literature. Furthermore, we provide applications to support our conclusions.

1. Introduction

The concept of fuzzy metric spaces was introduced by Kramosil and Michaleik [17] and later modified by George and Veeramani [7] with the aid of t-norms. Grabiec M. [10] established the fuzzy version of the Banach contraction principle, marking a significant milestone in the development of fixed point theory within fuzzy metric space [23].

Key Words and Phrases : *Fuzzy Metric Space, Fixed Point, Rational expression, and Fuzzy Point.*

Various authors have derived several fixed-point theorems in fuzzy metric spaces using concepts such as compatible maps, implicit relations, weakly compatible maps, and R-weakly compatible maps. Singh and Chouhan [21] and Cho [3] presented fixed point theorems in fuzzy metric spaces for four self-maps based on the concept of compatibility, which requires that two mappings be continuous.

Numerous researchers have established common fixed point theorems in fuzzy metric spaces under various contractive conditions. Notable contributions in this field include the works of Imdad Ali [15, 16], Chouhan S. et al. [1], Chouhan V.S. et al. [2], Dhinarvand [4], Eidi et al. [6], Kumar S. [18], Sneha et al. [25], Gopal et al. [9], Gupta V. et al. [11, 12, 13], Gupta S. et al. [14], Singh V. et al. [22], and Tiwari R. et al. [26]. In 2017, Govery A. and Singh M. [8] established a common fixed point theorem for six self-mappings in fuzzy metric spaces by employing the concepts of compatibility and weak compatibility. Recently, Sonam et al. [24] obtained fixed point results for soft fuzzy metric spaces. In 2024, Dubey H. et al. [5] proved common fixed point theorems for six and seven self-mappings in fuzzy metric spaces using the E.A. property.

Yadav and Wadhwa [28] conducted a study on fixed point theorems within fuzzy metric spaces, specifically focusing on -admissible mappings for both single-valued and set-valued functions. Recently, Tiwari S.K. and Agrawal R. [27] generalized and improved upon the results of Yonghong Shen et al. [29], building on concepts introduced by Schweizer and Sklar [23]. They established a common fixed point theorem in fuzzy metric spaces under a contractive condition.

This article aims to establish fixed-point results for rational-type contractive mappings in fuzzy metric spaces. Our findings generalize, improve, and extend the comparable results of Gupta et al. [11, 12, and 13] and Singh et al. [23], along with contributions from other researchers in the literature. Furthermore, we include applications to substantiate our findings.

2. Preliminaries Notes

In this section, we will present key definitions and results that will be referenced later.

Definition 2.1 ([7, 22]) : Let γ be any set. A fuzzy set A in γ is a function with domain γ and value in [0, 1].

Definition 2.2 ([19, 20]) : A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if it satisfies the following conditions:

(1) $*$ is commutative and associative,

(2) $*$ is continuous,

(3) $1 * a = a$ for all $a \in [0, 1]$,

(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0.1]$.

Definition 2.3 ([1]) : A fuzzy metric space is an ordered tripled $(\gamma, M, *)$ such that γ is a non- empty set, $*$ is continuous t-norm and M is a fuzzy set on $\gamma \times \gamma \times (0, \infty)$ satisfying the following conditions, for all $x, y.z \in \gamma, s, t > 0$

(i) $M(x, y, 0) > 0$;

(ii) $M(x, y, t) = 1$ iff $x = y$, for all $t \geq 0$;

(iii) $M(x, y, t) = M(y, x, t)$;

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s), \forall x, y, z \in \gamma$ and $t, s > 0$;

(v) $M(x, y.) : (0, \infty) \rightarrow (0, 1)$ is left continuous; and

(vi) $\lim_{t \to \infty} M(x, y, t) = 1 \ \ \forall \ \ x, y, \in \gamma$.

Definition 2.4([2]) : The triplet $(\gamma, M, *)$ is said to be fuzzy metric space, if γ is an arbitrary set, $*$ is continuous and M is fuzzy set on $\gamma^2 \times [0, \infty)$ satisfying the following conditions:

(i) $M(x, y, t) > 0$;

(ii) $M(x, y, t) = 1$ iff $x = y$, for all $t \geq 0$;

(iii) $M(x, y, t) = M(y, x, t)$;

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s), \ \forall \ \ x, y, z \in \gamma$ and $t, s > 0$;

(v) $M(x, y, .) : (0, \infty) \rightarrow [0, 1]$ is left continuous; and

(vi) $\lim_{t \to \infty} M9x, y, t) = 1, \ \ \forall \ \ x, y \in \gamma$.

Then M is called a fuzzy metric on γ and $M(x,t,t)$ denotes the degree of nearest between x and y with respect to t.

Definition 2.5 ([3]) : Let $(\gamma, M, *)$ be a fuzzy metric space, $x \in \gamma$ and $\{x_n\}$ be a sequence in γ. Then,

(1) $\{x_n\}$ is said to converge to x if $M(x_n, x, t) \to 1$ as $n \to \infty$, for all $t \geq 0$.

(2) $\{x_n\}$ is said to Cauchy sequence if $M(x_{n+p}, x_n, t) \to 1$ as $n \to \infty$ for every $t > 0$ and each positive integer p.

(3) $(X, M, *)$ is said to be complete fuzzy metric space if every Cauchy sequence is convergent in X.

Lemma 2.6 ([3]) $M(x, y, \cdot)$ is non-decreasing, for all $x, y \in X$.

Lemma 2.7 ([3]) : If there exist $k \in (0,1)$ such that $M(x, y, kt) \geq M(x, y, t)$, for all $x, y \in X$ at $t \in (0, \infty)$, then $x = y$.

3. Main Results

Our main results are as follows:

Theorem 3.1 : Let $(U, M, *)$ be a complete fuzzy metric space and $E : X \to X$ be a mapping satisfies

$$M(Ex, Ey, kt) \geq \min\left\{M(x,y,t), \frac{M(x,Ex,t)M(y,Ey,T)}{M(x,y,t)}, \frac{M(x,Ey,t)M(y,Ex,t)}{M(x,y,t)}\right\} \tag{3.3.1}$$

For all $x, y \in U$ and for some $k \in (0,1)$ then E has a unique fixed point.

Proof : Let $x_0 \in X$ and consider an iterative sequence $\{x_n\}$ in U such as

$$x_{n+1} = Ex_n, \quad \forall \quad n \in N.$$

Now, put x x_{n-1} and $y = x_n$ in equation (3.1.1), we have

$$\begin{aligned}
&M(x_{n,n+1}, t) = M(Ex_{n-1}, Ex_n, t)\\
&\geq \left\{ \min \left\{ \begin{array}{c} M\left(x_{n-1}, x_n, \frac{t}{k}\right), \frac{M(x_{n-1}, Ex_{n-1}, \frac{t}{k}) \cdot M(x_n, Ex_n, \frac{t}{k})}{M(x_{n-1}, x_n, \frac{t}{k})} \\ \\ \frac{M(x_{n-1}, Ex_n, \frac{t}{k}) \cdot M(x_n, Ex_{n-1}, \frac{t}{k})}{M(x_{n-1}, x_n, \frac{t}{k})} \end{array} \right\} \right\}\\
&= \left\{ \min \left\{ \begin{array}{c} M\left(x_{n-1}, x_n, \frac{t}{k}\right), \frac{M(x_{n-1}, Ex_n, \frac{t}{k}) \cdot M(x_n, x_{n+1}, \frac{t}{k})}{M(x_{n-1}, x_n, \frac{t}{k})} \\ \\ \frac{M(x_{n-1}, x_{n+1}, \frac{t}{k}) \cdot M(x_n, Ex_n, \frac{t}{k})}{M(x_{n-1}, x_n, \frac{t}{k})} \end{array} \right\} \right\}\\
&= \left\{ \min \left\{ \left(x_{n-1}, x_n, \frac{t}{k}\right), M\left(x_n, x_{n+1}, \frac{t}{k}\right) \right\} \right\}
\end{aligned}$$

So, we have

$$M(x_n, x_{n+1}, t) \geq \left\{ \min \left\{ M\left(x_{n-1}, x_n, \frac{t}{k}\right), M\left(x_n, x_{n+1}, \frac{t}{k}\right) \right\} \right\} \tag{3.1.2}$$

Following two cases arises.
Case -1 : If

$$\left\{ \min \left\{ M\left(x_{n-1}, x_n, \frac{t}{k}\right), M\left(x_n, x_{n+1}, \frac{t}{k}\right) \right\} \right\} = M\left(x_n, x_{n+1}, \frac{t}{k}\right).$$

Then from (3.1.2), we get

$$M(x_n, x_{n+1}, t) \geq M\left(x_n, x_{n+1}, \frac{t}{k}\right)$$

which is proved by Lemma 2.7.
Case-2 : If

$$\left\{ \min \left\{ M\left(x_{n-1}, x_n, \frac{t}{k}\right), M\left(x_n, x_{n+1}, \frac{t}{k}\right) \right\} \right\} = M\left(x_{n-1}, x_n, \frac{t}{k}\right).$$

Then from (3.1.2), we have

$$M(x_n, x_{n+1}, t) \geq M\left(x_{n-1}, x_n, \frac{t}{k}\right).$$

Now $M(x_n, x_{n+1}, t) \geq M(x_{n-1}, x_n, \frac{t}{k}$, then we have

$$\begin{aligned}
&= M\left(Ex_{n-2}, x_{n-1}, \frac{t}{k}\right) \\
&\geq M\left(x_{n-2}, x_{n-1}, \frac{t}{k^2}\right) \\
&\geq M\left(x_{n-1}, x_{n-2}, \frac{t}{k^3}\right) \\
&\vdots \\
&\geq M\left(x_0, x_1, \frac{t}{k^n}\right) \qquad (3.1.3)
\end{aligned}$$

Now for any positive integer $m \in N$, we have

$$\begin{aligned}
M(x_n, x_{n+m}, t) &\geq M\left(x_n, x_{n+1}, \frac{t}{p}\right) * M\left(x_{n+1}, x_{n+2}, \frac{t}{p^2}\right) + \cdots * \\
&\quad M\left(x_{n+m-1}, x_{n+m}, \frac{t}{p^n}\right). \\
&\geq M\left(x_0, x_1, \frac{t}{pk^n}\right) * M\left(x_0, x_1, \frac{t}{(p^2k^2)k^{n-1}}\right) * \cdots * \\
&\quad M\left(x_0, x_1, \frac{t}{p^m k^{n-1}}\right) \\
&\geq 1 * 1 * \cdots * 1 \\
&= 1.
\end{aligned}$$

Letting $n \to \infty$. Then we get

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1.$$

Hence $\{x_n\}$ is Cauchy sequence. Since $(U, M, *)$ is CFMS. So, there exists an element $u \in U$ such that

$$\lim_{n\to\infty} x_n = u.$$

Now, we prove that u is a fixed point of E. Now consider

$$\begin{aligned}
M(Eu, u, t) &\geq M(Eu, x_{n+1}, t) * M(x_{n+1}, u, t) \\
&= M(Eu, x_n, t) * M(x_{n+1}, x_n, t) \qquad (3.1.4)
\end{aligned}$$

Now,

$$
\begin{aligned}
M(Eu, Ex_n, t) \quad &\geq \quad \left\{ \begin{array}{c} \min\left(u, x_n \frac{t}{k}\right) \frac{M(x_n, Ex_n, \frac{t}{k}) * M(x_n, Ex_n, \frac{t}{k})}{M(u, x_n, \frac{t}{k})}, \\ \\ \frac{M(x_n, Ex_n, \frac{t}{k}) * M(x_n, Ex_n, \frac{t}{k})}{M(u, x_n, \frac{t}{k})} \end{array} \right\} \\
&= \min \left\{ \begin{array}{c} M\left(x_n, x_n, \frac{t}{k}\right), \frac{M(x_n, x_{n+1}, \frac{t}{k}) * M(x_n, x_{n+1}, \frac{t}{k})}{M(x_n, x_n, \frac{t}{k})} \\ \\ \frac{M(x_n, x_{n+1}, \frac{t}{k}) * M(x_n, x_{n+1}, \frac{t}{k})}{M(x_n, x_n, \frac{t}{k})} \end{array} \right\} \\
&= \min \left\{ \begin{array}{c} 1, M\left(x_n, x_{n+1}, \frac{t}{k}\right) * M\left(x_n, x_{n+1}, \frac{t}{k}\right), \\ M\left(x_n, x_{n+1}, \frac{t}{k}\right) * M\left(x_n, x_{n+1}, \frac{t}{k}\right) \end{array} \right\}.
\end{aligned}
$$

So, from (3.1.4), we get

$$
\begin{aligned}
M(Eu, u, t) \quad &= \quad \min\left\{ M\left(1, M\left(x_n, x_{n+1}, \frac{t}{k}\right) * M\left(x_{n+1}, x_n, \frac{t}{k}\right)\right)\right\} \\
&= \quad \min\{(1,1), *1\}, \\
&= \quad 1 * 1 \\
&= \quad 1.
\end{aligned}
$$

Thus $Eu = u$. Hence E is has fixed point.
Now, we prove uniqueness. For some $u^* \in U$ such that $Eu^* = u^*$, then

$$
\begin{aligned}
M(u^*, u, t) \quad &= \quad M(Eu^*, Eu, t) \\
&\geq \min \left\{ \begin{array}{c} M\left(u^*, u, \frac{t}{k}\right), \frac{M(u^*, Eu^*, \frac{t}{k}) * M(u, Eu, \frac{t}{k})}{M(u^*, u, \frac{t}{k})}, \\ \\ \frac{M(u^*, Eu, \frac{t}{k}) * M(u, Eu^*, \frac{t}{k})}{M(u^*, u, \frac{t}{k})} \end{array} \right\} \\
&= \min \left\{ \begin{array}{c} M\left(u^*, u, \frac{t}{k}\right), \frac{M(u^*, u^*, \frac{t}{k}) * M(u, u, \frac{t}{k})}{M(u^*, u, \frac{t}{k})}, \\ \\ \frac{M(u^*, u, \frac{t}{k}) * M(u, u^*, \frac{t}{k})}{M(u^*, u, \frac{t}{k})} \end{array} \right\} \\
&= \min\left\{ M\left(u^*, u, \frac{t}{k}\right), 1 \right\}.
\end{aligned}
$$

If $\min\left\{M\left(u^*, u, \frac{t}{k}\right), 1\right\} = 1$. Then $M(u^*, u, t) = 1$. Thus $u^* = u$.
If $\min\left\{M\left(u^*, u, \frac{t}{k}\right), 1\right\} = M\left(u^*, u, \frac{t}{k}\right)$, then

$$M(u^*, u, t) \geq M(u^*, u, t).$$

So, from Lemma 2.7, we get $u^* = u$. Hence fixed point is unique.
Remark 3.2 : If we take

$$\min\left\{M(x,y,t), \frac{M(x,Ex,t)M(x,Ey,t)}{M(x,y,t)}, \frac{M(x,Ey,t)M(y,Ex,t)}{M(x,y,t)}\right\}$$
$$= M(x,y,t)$$

in theorem 3.1, then we get the result of Grabiec, M.(1988) as follows Corollary 3.3.
Corollary 3.3 : Let $(X, M, *)$ be a complete fuzzy metric space and $E : X \to X$ be a mapping satisfies

$$M(Ex, Ey, kt) \geq M(x, y, t)$$

for all $x, y \in X$ and for some $k \in (0,1)$. Then E has a unique fixed point.
The following example is furnished to illustrate Theorem 3.1.
Example 3.4 : Let $U = \{0, 1, 2\}$. A function $M : U \times U \times [0,\infty) \to [0,1)$ defined by

$$M(x,y,t) = \frac{t}{t+(x-y)^2}, \quad \forall\ x,y \in U.$$

Then $(U, M, *)$ is a complete FMS. Define a mapping $E : U \to U$ such as

$$Ex = \sqrt{2x}.$$

Now, for all $t > 0$, from 3.1.1 we have

$$M(\sqrt{2x}, \sqrt{2y}, kt) \geq \min\left\{ M(x,y,t), \begin{array}{c} \frac{M(x,\sqrt{2x},t)*M(y,\sqrt{2y},t)}{M(x,yt)}, \\ \frac{M(x,\sqrt{2y},t)*M(y,\sqrt{2x},t)}{M(x,yt)} \end{array} \right\}$$

Therefore,

$$\frac{kt}{kt+(\sqrt{2x}-\sqrt{2y})^2} \geq \min\left\{ \begin{array}{c} \frac{t}{t+(x-y)^2}, \frac{t}{t+(x-\sqrt{2x})^2} * \frac{t}{t+(y-\sqrt{2y})^2}, \\ \frac{t}{t+(x-\sqrt{2y})^2} * \frac{t}{t+(y-\sqrt{2x})^2} \end{array} \right\}$$

Or

$$\frac{t}{t+(x-y)^2} \geq \min\left\{ \begin{array}{c} \frac{t}{t+(x-y)^2}, \frac{t}{t+x^2(x-\sqrt{2})^2} * \frac{t}{t+y^2(y-\sqrt{2})^2}. \\ \frac{t}{t+(x-\sqrt{2y})^2} + \frac{t}{t+(y-\sqrt{2x})^2} \end{array} \right\}. \tag{3.1.5}$$

If

$$\min\left\{\begin{array}{c}\frac{t}{t(+x-y)^2},\frac{t}{t+x^2(x-\sqrt{2})^2}*\frac{t}{t+y^2(y-\sqrt{2})^2},\\ \frac{t}{t+(x-\sqrt{2y})^2}*\frac{t}{t+(y-\sqrt{2x})^2}\end{array}\right\}$$

$$=\min\left\{\begin{array}{c}\frac{t}{t+x^2(x-\sqrt{2})^2}*\frac{t}{t+y^2(y-\sqrt{2})^2},\\ \frac{t}{t+(x-\sqrt{2y})^2},*\frac{t}{t+(y-\sqrt{2x})^2}\end{array}\right\}$$

then from (3.1.5)

$$\frac{t}{t+(x-y)^2}\geq\min\left\{\begin{array}{c}\frac{t}{t+x^2(x-\sqrt{2})^2}*\frac{t}{t+y^2(y-\sqrt{2})^2},\\ \frac{t}{t+(x-\sqrt{2y})^2}*\frac{t}{t+(y-\sqrt{2x})^2}\end{array}\right\}$$

$$\Rightarrow M(Ex,Ey,kt)\geq\min\left\{M(x,y,t),\frac{M(x,Ex,t)M(x,Ex,t)}{M(x,y,t)},\frac{M(x,Ey,t)M(y,Ex,t)}{M(x,y,t)}\right\}.$$

If

$$\min\left\{\begin{array}{c}\frac{t}{t+(x-y)^2},\frac{t}{t+x^2(x-\sqrt{2})^2}*\frac{t}{t+y^2(y-\sqrt{2})^2},\\ \frac{t}{t+(x-\sqrt{2y})^2}*\frac{t}{t+(y-\sqrt{2x})^2}\end{array}\right\}=\frac{t}{t+(x-y)^2},$$

then from (3.1.5) we get

$$\frac{t}{t+(x-y)^2}=\frac{t}{t+(x-y)^2}.$$

$$\Rightarrow M(Ex,Ey,kt)\geq\min\left\{M(x,y,t),\frac{M(x,Ex,t)M(x,Ex,t)}{M(x,y,t)},\frac{M(x,Ey,t)M(y,Ex,t)}{M(x,y,t)}\right\}.$$

Thus all conditions of Theorem 3.1 are satisfied as $Ex=\sqrt{2x}$

$$\Rightarrow E0=0.$$

Thus E has a unique fixed point $E=0$.

Theorem 3.5 : Let $(U,M,*)$ be a complete fuzzy metric space and $E:X\to X$ be a mapping satisfies

$$M(Ex,Ey,kt)\geq\min\left\{\begin{array}{c}M(x,y,t),\frac{M(y,Ey,t)[1+M(x,Ex,t)]}{1+M(x,y,t)}\\ \frac{M(x,Ex,t)[1+M(y,Ey,t)]}{1+M(Ex,Ey,t)},\frac{M(y,Ex,t)[1+M(x,Ey,t)]}{1+M(x,y,t)}\end{array}\right\}\qquad(3.5.1)$$

For all $x,y,\in U$ and for some $k\in(0,1)$, then E has a unique fixed point.

Proof : Let $x_0 \in X$ and consider an iterative sequence $\{x_n\}$ in U such as

$$x_{n+1} = Ex_n, \quad \forall \ n \in N.$$

Now, put $x = x_{n-1}$ and $y = x_n$ in equation (3.5.1), we get

$$\begin{aligned} M(x_n, x_{n+1}, t) &= M(Ex_{n-1}, Ex_n, t) \\ &\geq \left\{ \min \left\{ \begin{array}{c} M\left(x_{n-1}, x_n, \frac{t}{k}\right), \frac{M(x_n, Ex_n, \frac{t}{k})[1+M(x_{n-1}, Ex_{n-1}, \frac{t}{k})]}{1+M(x_{n-1}, x_n, \frac{t}{k})}, \\ \frac{M(x_{n-1}, Ex_{n-1}, \frac{t}{k})\cdot[1+M(x_n, Ex_n, \frac{t}{k})]}{1+M(Ex_{n-1}, Ex_n, \frac{t}{k})}, \\ \frac{M(x_n, Ex_{n-1}, \frac{t}{k})[1+M(x_{n-1}, Ex_n, \frac{t}{k})]}{1+M(x_{n-1}, x_n, \frac{t}{k})} \end{array} \right\} \right\} \\ &= \left\{ \min \left\{ \begin{array}{c} M(x_{n-1}, x_n, \frac{t}{k}), \frac{M(x_n, x_{n+1}, \frac{t}{k})[1+M(x_{n-1}, x_n, \frac{t}{k})]}{1+M(x_{n-1}, x_n, \frac{t}{k})}, \\ \frac{M(x_{n-1}, x_n, \frac{t}{k})\cdot[1+M(x_n, x_{n+1}, \frac{t}{k})]}{M(x_n, x_{n+1}, \frac{t}{k})}, \\ \frac{M(x_n, x_n, \frac{t}{k})[1+M(x_{n-1}, x_{n+1}, \frac{t}{k})]}{1+M(x_{n-1}, x_n, \frac{t}{k})} \end{array} \right\} \right\} \\ &= \left\{ \min \left\{ M\left(x_{n-1}, x_n, \frac{t}{k}\right), M\left(x_n, x_{n+1}, \frac{t}{k}\right) \right\} \right\}. \end{aligned}$$

So, we have

$$M(x_n, x_{n+1}, t) \geq \left\{ \min \left\{ M\left(x_{n-1}, x_n, \frac{t}{k}\right), M\left(x_n, x_{n+1}, \frac{t}{k}\right) \right\} \right\}. \tag{3.5.2}$$

Following two cases arises and one can complete the proof using the same procedure after inequality (3.1.2) as in theorem 3.1.
To prove the above both Theorems as follows, we use a control function $r : [0, 1) \to [0, 1)$ which is continuous and non decreasing such as

1. $r(0 = 0, r(1) = 1$.
2. $r(a) > \alpha$ for $0 < \alpha < 1$.

Theorem 3.6 : Let $(U, M, *)$ be a complete fuzzy metric space and $E : X \to X$ be a mapping satisfies

$$M(Ex, Ey, kt) \geq r \left\{ \min \left\{ \begin{array}{c} M(x, y, t), \frac{M(x,Ex,t)M(y,Ey,t)}{M(x,y,t)}, \\ \frac{M9x,Ey,t)M(y,Ex,t)}{M(x,y,t)} \end{array} \right\} \right\}. \tag{3.6.1}$$

for all $x, y \in U$ and for some $k \in (0, 1)$ then E has a unique fixed point.

Proof : Since r is a continuous function and $r(\alpha) > \alpha$ for $0 < \lambda <!$, then from (3.6.1), using proof of theorem 3.1, we get required result.
Theorem 3.7 : Let $(U, M, *)$ be a complete fuzzy metric space and $E : X \to X$ .. be a mapping satisfies

$$M(Ex, Ey, kt) \geq r\left\{\min\left\{\begin{array}{c} M(,y,t), \frac{M(y,Ey,t)[1+M(x,Ex,t)]}{1+M(x,y,t)}, \\ \frac{M(x,Ex,t)[1+M(y,Ey,t)]}{1+M(Ex,Ey,t)}, \\ \frac{M(y,Ex,t)[1+M(x,Ey,t)]}{1+M(x,y,t)} \end{array}\right\}\right\} \quad (3.7.1)$$

for all $x, y \in U$ and for some $k \in (0, 1)$ then E has a unique fixed point.
Proof : Since r is a continuous function and $r(\alpha) > \alpha$ for $0 < \alpha < 1$. Then from (3.7.1), using proof of theorem 3.5, we get required result.

4. Applications

In this section, we applying our recent results to integral-type functions involving rational contraction is the main objective of this section. To this effect, we indicate Ψ the set of all functions $\Psi : [0, \infty) \to [0, \infty)$ which satisfy the following hypothesis:

(i) each Ψ must be a Lebesgue-integrable mapping on every compact subset of $[0, \infty)$;

(ii) We have $\Psi(t) = \int_0^t \varphi(t)dt$ for any be a non decreasing and continuous function, moreover, for each $\epsilon > 0, \varphi(\epsilon) > 0$. It is also implies that$\varphi(t) = 0$ iff $t = 0$.

Theorem 4.1 : Let $(U, M, *)$ be a complete fuzzy metric space and $E : X \to X$ be a mapping satisfies

$$\int_0^{M(Ex,Ey,kt)} \varphi(t)dt \geq \min\left\{\begin{array}{c} \int_0^{M(x,y,t)} \varphi(t)dt, \int_0^{\frac{M(x,Ex,t)M(y,Ey,t)}{M(x,y,t)}} \varphi(t)dt, \\ \int_0^{\frac{M(x,Ey,t)M(y,Ex,t)}{M(x,y,t)}} \varphi(t)dt \end{array}\right\}$$

for all $x, y \in U$ and for some $k \in (0, 1)$ then E has a unique fixed point. If

$$\min\left\{\begin{array}{c} \int_0^{M(x,y,t)} \varphi(t)dt, \int_0^{\frac{M(x,Ex,t)M(y,Ey,t)}{M(x,y,t)}} \varphi(t)dt, \\ \int_0^{\frac{M(x,Ey,t)M(y,Ex,t)}{M(x,y,t)}} \varphi(t)dt \end{array}\right\} = \int_0^{M(x,y,t)} \varphi(t)dt,$$

then we get the result as follows Corollary 3.3.
Corollary 4.2 : Let $(X, M, *)$ be a complete fuzzy metric space and $E : X \to X$ be a mapping satisfies

$$\int_0^{M(Ex,Ey,kt)} \varphi(t)dt = \int_0^{M(x,y,t)} \varphi(t)dt,$$

for all $x, y \in X$ and for some $k \in (0,1)$. Then E has a unique fixed point.
Theorem 4.3 : Let $(U, M, *)$ be a complete fuzzy metric space and $E : X \to X$ be a mapping satisfies

$$\int_0^{M(Ex,Ey,kt)} \varphi(t)dt \geq \min \left\{ \begin{array}{c} \int_0^{M(x,y,t)} \varphi(t)dt, \frac{M(y,Ey,t)[1+M(x,Ex,t)]}{M(x,y,t)} \varphi(t)dt, \\ \int_0^{\frac{M(x,Ex,t)[1+M(y,Ey,t)]}{M(Ex,Ey,t)}} \varphi(t)dt, \\ \int_0^{\frac{M(y,Ex,t)[1+M(x,Ey,t)]}{M(x,y,t)}} \varphi(t)dt \end{array} \right\}$$

for all $x, y \in U$ and for some $k \in (0,1)$ then E has a unique fixed point.
Theorem 4.4 : Let $(U, M, *)$ be a complete fuzzy metric space and $E : X \to X$ be a mapping satisfies

$$\int_0^{(M(Ex,Ey,kt)} \varphi(t)dt \geq \left\{ r \left\{ \min \left\{ \begin{array}{c} M(x,y,t), \frac{M(x,Ex,t)M(y,Ey,t)}{M(x,y,t)}, \\ \frac{M(x,Ey,t)M(y,Ex,t)}{M(x,y,t)} \end{array} \right\} \right\} \right\}$$

for all $x, y \in U$ and for some $k \in (0,1)$ then E has a unique fixed point.
Theorem 4.3 : Let $(U, M, *)$ be a complete fuzzy metric space and $E : X \to X$ be a mapping satisfies

$$\int_0^{M(Ex,Ey,kt)} \varphi(t)dt$$
$$\geq \left\{ r \left\{ \min \left\{ \begin{array}{c} \int_0^{M(x,y,t)} \varphi(t)dt, \frac{M(y,Ey,t)[1+M(x,Ex,t)]}{M(x,y,t)} \varphi(t)dt, \\ \int_0^{\frac{M(x,Ex,t)[1+M(y,Ey,t)]}{M(Ex,Ey,t)}} \varphi(t)dt, \\ \int_0^{\frac{M(y,Ex,t)[1+M(x,Ey,t)]}{M(x,y,t)}} \varphi(t)dt \end{array} \right\} \right\} \right\}$$

for all $x, y \in U$ and for some $k \in (0,1)$ then E has a unique fixed point.

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